e content for students of patliputra university

B. Sc. (Honrs) Part 1paper 2

Subject:Mathematics

Title/Heading of topic:Solved problems on Euler's theorem

By Dr. Hari kant singh

Associate professor in mathematics

Rrs college mokama patna

❖ Solved Problems on Euler's Theorem : -

Q. If $u=\frac{x^3y^3z^3}{x^3+y^3+z^3}+\log\left\{\frac{xy+yz+xz}{x^2+y^2+z^2}\right\}$

Find $x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$

Let u=v+w

$V(tx,ty,tz)=\frac{t9}{t3}v(x,y,z)$

$=t^6\ v(x,y,z)$

∴ v is homogenous function of degree 6.

∴ EuLer theorem for v we can write

$x.\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=6v$ (i)

$w(tx,ty,tz)=t^0\ w(x,y,z)$

∴ w is homogenous function of degree o

Hence by Euler theorem for w

$x.\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=0w=0$ (ii)

Equation (i) +(ii) we get ;

$x\left\{\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right\}+y\left\{\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right\}+z\left\{\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right\}=6v$

$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=6v$

$=6\frac{x^3y^3z^3}{x^3+y^3+z^3}$

Q. $u=\frac{f(0)}{r}$, $x=r\cos\theta$

$Y=r\sin\theta$

P.T $x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-u$

$u=\frac{f\left(\tan^{-1}\left(\frac{y}{x}\right)\right)}{\sqrt{x^2+y^2}}$

$u(xt,yt)=\frac{f\left(\tan^{-1}\left(\frac{y}{x}\right)\right)}{\sqrt{x^2+y^2}}=t^{-1}u(x,y)$

u is homogenous of degree(-1)

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-u$

Q. $u = \frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}$

$$u(xt,yt) = \frac{1}{x^2t^2}+\frac{1}{xt+yt}+\frac{\log\left(\frac{x}{y}\right)}{x^2+y^2+2}$$

$$=\frac{1}{t^2}\left\{\frac{1}{x^2}+\frac{1}{xy}+\frac{\log\left(\frac{x}{y}\right)}{x^2+y^2+2}\right\}$$

$$=t^{-2}\left\{\frac{1}{x^2}+\frac{1}{xy}+\frac{\log\left(\frac{x}{y}\right)}{x^2+y^2+2}\right\}=t^{-2}u(x,y)$$

∴ u is homogenous function of degree −2

∴ by Euler's theorem for u,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(-2)u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2u=0$$

u=f(v)

∴ prove that

∴ prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nvf'(v)$$

Deduce that if u=logv

Then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n$

Since v is homogenous function of degree n, by Euler's theorem for v

$$\therefore \quad x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv \qquad (*)$$

But u=f(v)

$u\longrightarrow v\longrightarrow xy$

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}.\frac{\partial v}{\partial x}=f'(v).\frac{\partial v}{\partial x} \qquad (i)$$

And

(i)+(ii)

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=4v-\frac{1}{2}\cot w$

$=4\frac{x^2y^2z^2}{x^2+y^2+z^2}-\frac{1}{2}\cot\left\{\cos^{-1}\left\{\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right\}\right\}$

Q. If $u=\log(x^3+y^3-x^2y-xy^2)$

P.T. (i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-3$

Solution:

$u=\log(x^3+y^3-x^2y-xy^2)$

$f_1(u)=e^u=x^3+y^3-x^2y-xy^2=f(x,y)$

Then $f(xt,yt)=t^3f(x,y)$

∴ f is a homogenous function of degree 3.

And $f=f_1(u)$

∴ by corollary (ii)

And

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial y}.\frac{\partial v}{\partial y}=f'(v).\frac{\partial v}{\partial y} \qquad \text{(ii)}$$

(i)× x +(ii) × y

$$\therefore \quad x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=f'(v)\left\{x\ \frac{\partial v}{\partial x}\ +y\frac{\partial v}{\partial y}\right\}$$

$$=f'(v)nv \qquad \text{(from *)}$$

Now if u=logv=f(v)

$$\therefore f'(v)=\frac{1}{v}$$

$$\therefore x\ \frac{\partial u}{\partial x}\ +y\frac{\partial u}{\partial y}=nv\times\frac{1}{v}=n$$

Hence proved

Q. Let $v=\frac{x^2y^2z^2}{x^2+y^2+z^2}$ $\qquad$ $w=\cos^{-1}\left\{\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right\}$

$\therefore$ u=v+w

$V(xt,yt,zt)=\frac{t^6}{t^2}v(x,y,z)$

$=t^4\ v(x,y,z)$

$\therefore$ v is homogenous function of degree 4.

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial y}=4v \qquad \text{(i)}$$

now,

$$w=\cos^{-1}\left\{\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right\}$$

$$f_1(w)=\cos w=\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}=f(x,y,z)$$

$$F(xt,yt,zt)=t^{\frac{1}{2}}f(x,y,z)$$

$\therefore$ f is a homogenous function of degree $\frac{1}{2}$ and $f=f_{1(w)}$

$\therefore$ by corollary (ii)

$$x.\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=\frac{1}{2}\frac{f_1(w)}{f_1'(w)}$$

$$=\frac{1}{2}\left\{\frac{\cos w}{-\sin w}\right\}=-\frac{1}{2}\cot w \qquad \text{(ii)}$$

∴ f is a homogenous function of degree 3.

And $f=f_1(u)$

∴ by corollary (ii)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{3f_1(u)}{f_1'(u)}=\frac{3e^u}{e^u}=3$$

by corollary (iii)

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)(g'(u)-1)$$

$$g(u)=\frac{nf(u)}{f'(u)}=3$$

$$g'(u)=0$$

∴ L.H.S of (1)=3 (0−1)=−3

Q. P.T. $\sin 2u(1-4\sin^2 u)$

$$u=\tan^{-1}\left\{\frac{x^3+y^3}{x-y}\right\}$$

solution:

$$f_{,}(u) = \tan au = \frac{x^3+y^3}{x-y} = f(x,y)$$

f(xt,yt)=$t^2 f(x,y)$

∴ u is homogenous function of degree 2 by corollary 3.

$$x^2u_{xx} + 2xyu_{xx} + y^2u_{yy} = g(u)(\,g'(u) - 1) \quad \text{(i)}$$

Where g(u)=$\frac{2f(u)}{f'(u)}$

g(u)=$\frac{2\tan u}{\sec^2 u} = \frac{2\sin u}{\cos u} \times \cos^2 u = 2\sin u.\cos u = \sin_2 u$

L.H.S of (i) =sin2u(2cos2u−1) (ii)

=sin2u(2(1−2$\sin^2 u) - 1$)

$=\sin 2u(1-4\sin^2 u)$ ans3

From (ii) or $=2\sin 2u.\cos 2u-\sin 2u$

$=\sin 4u-\sin 2u$ ans2

From (iii) or =using sin(−sinu) of ans2

$=2\cos 3u.\sin u$ ans1

Q. $u=\text{cosec} = \sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}=f(x,y)$

$f(xt,yt)=t^{1/12}f(x,y)$

f is a homogenous function of degree $\frac{1}{2}$

$f=f_{,}(u)$

by corollary (3)

Q. If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$ (i)

Where u is homogenous function of degree and in x,y,z

P.T.

$u_x{}^2+u_y{}^2+u_z{}^2=2nu=2(xu_x+yu_y+u_z)$

Solution:

Differentiate (i) w.r.t. x

$$\frac{-x^2}{(a^2+u)^2}u_x+\frac{2x}{a^2+u}-\frac{-y^2}{b^2+u}u_x-\frac{z^2}{c^2+u}u_x=0$$

$$\frac{2x}{a^2+u}=ux\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}$$

$$\therefore \frac{2x}{(a^2+u)}=uxf \text{ where. } f=\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}$$

Similarly

$$\frac{2y}{(b^2+u)} = uyf \qquad (3)$$

$$\frac{2z}{(c^2+u)} = uzf \qquad (4)$$

From (2),(3),(4)

$$ux^2+uy^2+uz^2 = \frac{4x^2}{f^2(a^2+u)^2} + \frac{4y^2}{f^2(b^2+u)^2} + \frac{4z^2}{f^2(c^2+u)^2}$$

$$= \frac{4}{f^2}\left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}$$

$$= \frac{4}{f^2} \times f$$

$$= \frac{4}{f}$$

$$ux^2+uy^2+uz^2 = \frac{4}{f} \qquad (5)$$

∵ u is a homogenous function of deg n

∴ By Enter theorem xyx+ yuy+ zuz = nu (6)

From (s) & (6)

$\frac{4}{f}$=2nu   i.e. $\frac{2}{f}$=nu

$$\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}=f(xu_x+yu_y+zu_z)$$

$$2\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}=f(nu)$$

$$2=fnu$$

$$\frac{2}{f}=nu$$

Hence proved